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Short derivation of Feynman Lagrangian for general diffusion processes

F Langouche,†§ D Roekaerts‡§ and E Tirapegui||

§ Instituut voor Theoretische Fysica, KU Leuven, B-3030 Leuven, Belgium
|| Institut de Physique Théorique, UCL, B-1348 Louvain-la-Neuve, Belgium

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Abstract. A short derivation of the Feynman Lagrangian for general diffusion processes is given by a technique relying on the use of different discretisations which are related by equivalence relations under the n-dimensional integral whose limit is the path integral. In this way calculation of the differential equation satisfied by the path integral is avoided.

We consider, in the m-dimensional space $\boldsymbol{q}=(q^1,\ldots,q^m)$, the most general differential equation of the form $\dot{P}(\boldsymbol{q},t)=\mathscr{L}(\boldsymbol{\partial},\boldsymbol{q},t)P(\boldsymbol{q},t)$, where $\partial_\mu=\partial/\partial q^\mu$ and only derivatives up to second order are allowed in the operator $\mathscr{L}$. We write it as

$$\dot{P}(\boldsymbol{q},t)=[\partial_\mu(A^\mu(\boldsymbol{q},t)+\tfrac{1}{2}\partial_\nu D^{\mu\nu}(\boldsymbol{q},t))+V(\boldsymbol{q},t)]P(\boldsymbol{q},t), \tag{1}$$

(summation over repeated indices is to be understood), where we take $D^{\mu\nu}(\boldsymbol{q},t)$, which is symmetric in its two indices, to be positive definite, and consequently $D(\boldsymbol{q},t)=\det D^{\mu\nu}(\boldsymbol{q},t)\neq 0$. For $V(\boldsymbol{q},t)=0$ one has the usual Fokker–Planck equation. If we assume that $P(\boldsymbol{q},t)$ transforms as $\bar{P}(\bar{\boldsymbol{q}},t)=|\det(\partial(\bar{q})/\partial(q))|^{-1}P(\boldsymbol{q},t)$ under a general coordinate transformation $\bar{q}^\mu=\bar{q}^\mu(\boldsymbol{q})$, which is the case for the Fokker–Planck equation where $P(\boldsymbol{q},t)$ is the probability density, then the transformation properties of (1) can be studied (Graham 1977b) in detail and it is convenient to introduce a Riemann space structure in $\boldsymbol{q}$ space using $D^{\mu\nu}(\boldsymbol{q},t)$ as the contravariant metric tensor, this choice being dictated by the transformation of this last quantity.

We are interested in the solution $P(\boldsymbol{Q},T;\boldsymbol{Q}_0,t_0)$ of (1) which reduces to $\delta(\boldsymbol{Q}-\boldsymbol{Q}_0)$ for $t=t_0$. It is convenient to introduce an operator formalism (Langouche *et al* 1979a), writing $P(\boldsymbol{Q},T;\boldsymbol{Q}_0,t_0)=\langle\boldsymbol{Q}|U(T,t_0)|\boldsymbol{Q}_0\rangle$ with $\mathrm{i}\partial_tU(t,t')=\hat{H}(\hat{p}_\mu,\hat{q}^\mu,t)U(t,t')$ ($\partial_t\equiv\partial/\partial t$) and $\hat{H}=-\tfrac{1}{2}\mathrm{i}\hat{p}_\mu\hat{p}_\nu D^{\mu\nu}(\hat{\boldsymbol{q}},t)-\hat{p}_\mu A^\mu(\hat{\boldsymbol{q}},t)+\mathrm{i}V(\hat{\boldsymbol{q}},t)$, where we have used the usual quantum mechanical notation and $\hat{p}_\mu=-\mathrm{i}\partial/\partial q^\mu$ in the $|\boldsymbol{q}\rangle$ basis. As has been remarked (Langouche *et al* 1979a,b, Leschke and Schmutz 1977), and especially by Dowker (1976), $P(\boldsymbol{Q},t;\boldsymbol{Q}_0,t_0)$ admits an infinity of functional integral representations

$$P(\boldsymbol{Q},t;\boldsymbol{Q}_0,t_0)=\int_{\gamma\ \boldsymbol{q}(t_0)=\boldsymbol{Q}_0}^{\boldsymbol{q}(t)=\boldsymbol{Q}} D\mu[\boldsymbol{q}]\exp\left[-\int_{t_0}^{t}d\tau L^\gamma(q,\dot{q},\tau)\right] \tag{2}$$

where γ stands for the discretisation involved in the definition of the functional integral and $L^\gamma=\tfrac{1}{2}D_{\mu\nu}(\boldsymbol{q},\tau)\dot{q}^\mu\dot{q}^\nu+a^\gamma_\mu(\boldsymbol{q},\tau)\dot{q}^\mu+b^\gamma(\boldsymbol{q},\tau)$ which we call the Lagrangian, depends

† Onderzoeker IIKW, Belgium.
‡ Aspirant NFWO, Belgium.

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explicitly on the discretisation γ through a^{γ}_{μ} and b^{γ}. The different discretisations γ arise naturally from the different orderings in $\hat{H}$ of the noncommuting operators $\hat{p}_{\mu}$ and $\hat{q}^{\nu}$ (Langouche *et al* 1979, Leschke and Schmutz 1977). The representation corresponding to discretisation in the prepoint (which we call $\gamma_1(0)$ in Langouche *et al* (1979b) and leads to the Lagrangian $L^{\gamma_1(0)}$) is most easily obtained since it corresponds to $\hat{H}$ ordered in the original anti-standard form given above. A standard calculation (Faddeev 1976, Langouche *et al* 1979a,b) using

$$\langle \boldsymbol{Q}|U(T,t_0)|\boldsymbol{Q}_0\rangle = \int \prod_{i=1}^{n} \mathrm{d}\boldsymbol{q}_i \prod_{i=0}^{n} \langle \boldsymbol{q}_{j+1}|U(t_{j+1},t_j)|\boldsymbol{q}_j\rangle,$$

with $U(t_{j+1},t_j) = \hat{I} - \mathrm{i}\epsilon\hat{H}(t_j)$, where $t_j = t_0 + j\epsilon$, $\epsilon = (T-t_0)/(n+1)$, allows us to express P as

$$\begin{aligned} P(\boldsymbol{Q},T;\boldsymbol{Q}_0,t_0) &= \lim_{n\to\infty} \int \prod_{i=1}^{n} \mathrm{d}\boldsymbol{q}_i \prod_{j=0}^{n} ND(\boldsymbol{q}_j,t_j)^{-1/2} \\ &\times \exp[-\tfrac{1}{2}\epsilon D_{\mu\nu}(\boldsymbol{q}_j,t_j)(\Delta^{\mu}_j/\epsilon + A^{\mu}(\boldsymbol{q}_j,t_j))(\Delta^{\nu}_j/\epsilon + A^{\nu}(\boldsymbol{q}_j,t_j)) + \epsilon\nabla(\boldsymbol{q}_j,t_j)], \\ N &= (2\pi\epsilon)^{-m/2},\ \boldsymbol{q}_0 = \boldsymbol{Q}_0,\ \boldsymbol{q}_{n+1} = \boldsymbol{Q} \end{aligned} \tag{3}$$

where $\boldsymbol{\Delta}_j = \boldsymbol{q}_{j+1} - \boldsymbol{q}_j$, $D = \det D^{\mu\nu}$, $D_{\mu\nu}D^{\nu\rho} = \delta^{\rho}_{\mu}$, and we have kept terms up to $\mathrm{O}(\epsilon)$ since the others do not contribute when $n\to\infty$.

On the other hand Feynman's definition of the functional integral representing P is (Cheng 1972, Dowker 1975)

$$\begin{aligned} P(\boldsymbol{Q},T;\boldsymbol{Q}_0,t_0) &= \lim_{n\to\infty} \int \prod_{i=1}^{n} \mathrm{d}\boldsymbol{q}_i \prod_{j=0}^{n} ND(\boldsymbol{q}_j + \Delta_j, t_j + \epsilon/2)^{-1/2} \\ &\times \exp\left[-\mathrm{stat}\int_{t_j}^{t_j+\epsilon} \mathrm{d}\tau\, L^{\mathrm{F}}(\boldsymbol{q}(\tau),\dot{\boldsymbol{q}}(\tau),\tau)\right], \end{aligned} \tag{4}$$

where stat indicates that the action $\int \mathrm{d}\tau\, L^{\mathrm{F}}$ is to be evaluated along the trajectory $\boldsymbol{q}(\tau)$ that makes this action stationary for $\boldsymbol{q}(t_j) = \boldsymbol{q}_j$, $\boldsymbol{q}(t_j+\epsilon) = \boldsymbol{q}_{j+1}$. The definition (4) determines the Feynman Lagrangian L^{F} which we want to calculate. The simplest way to do this is to write L^{F} in the general form $L^{\mathrm{F}} = \frac{1}{2}D_{\mu\nu}\dot{q}^{\mu}\dot{q}^{n} + a^{\mathrm{F}}_{\mu}(\boldsymbol{q},\tau)\dot{q}^{\mu} + b^{\mathrm{F}}(\boldsymbol{q},\tau)$, and then determine a^{F}_{μ} and b^{F} comparing (3) and (4) which represent the same quantity. The standard method (Cheng 1972, Dowker 1975) consists of determining the L^{F} such that (4) satisfies (1); the calculations are quite involved and the most elegant version is Dowker's, using normal coordinates (Dowker 1975). Both (3) and (4) are of the form $\int \Pi\, \mathrm{d}\boldsymbol{q}_i\, \Pi P_{\alpha}(\boldsymbol{q}_j + \boldsymbol{\Delta}_j, t_j+\epsilon; \boldsymbol{q}_j, t_j)$, $\alpha = 1, 2$, where P_1 can be read from (3) and P_2 from (4). In order to compute P_2 and compare with P_1, we calculate the action in (4) as a power series in $\boldsymbol{\Delta}$ and ϵ, keeping terms up to order ϵ, taking into account that $\Delta\Delta = \mathrm{O}(\epsilon)$, as can be seen from (3) (De Witt 1957, Langouche *et al* 1979b). We then want $\mathscr{A} = \int_t^{t+\epsilon} \mathrm{d}\tau\, L^{\mathrm{F}}(\boldsymbol{q}(\tau),\dot{\boldsymbol{q}}(\tau),\tau)$ for the trajectory $\boldsymbol{q}(\tau)$ solution of the Euler–Lagrange equations of L^{F} with $\boldsymbol{q}(t) = \boldsymbol{q}$, $\boldsymbol{q}(t+\epsilon) = \boldsymbol{q} + \boldsymbol{\Delta}$. Developing $D_{\mu\nu}(\boldsymbol{q}(\tau),\tau) = D_{\mu\nu}(\boldsymbol{q}(\tau),t) + \partial_t D_{\mu\nu}(\boldsymbol{q}(\tau),t)\,.\,(\tau - t) + \mathrm{O}((\tau-t)^2)$ we write $L^{\mathrm{F}} = L^0 + \Delta L$, $L^0 = \frac{1}{2}D_{\mu\nu}(\boldsymbol{q}(\tau),t)\dot{q}^{\mu}\dot{q}^{\nu}$. The calculation of $\int \mathrm{d}\tau\, \Delta L$ is direct (since it is independent of the differential equation obeyed by the trajectory):

$$\int_t^{t+\epsilon} \mathrm{d}\tau\, \Delta L = (\tfrac{1}{4}\partial_t D_{\mu\nu}\Delta^\mu\Delta^\nu + a^{\mathrm{F}}_\mu\Delta^\mu + \tfrac{1}{2}\partial_\nu a^{\mathrm{F}}_\mu\Delta^\mu\Delta^\nu + \epsilon b^{\mathrm{F}})_{(\boldsymbol{q},t)} + \mathrm{O}(\epsilon^{3/2}), \tag{5}$$

and $(\boldsymbol{q}, t)$ at the end of the formula means that all functions are evaluated at the prepoint $(\boldsymbol{q}, t)$. The contribution of L^0 to $\mathscr{A}$ is now given by the same calculation as in appendix A of Cheng (1972) which is simple and straightforward:

$$\int_t^{t+\epsilon} \mathrm{d}\tau\, L^0 = \bigg(\frac{1}{2\epsilon} D_{\mu\nu}\Delta^\mu\Delta^\nu + \frac{1}{4\epsilon}\partial_\mu D_{\nu\rho}\Delta^\mu\Delta^\nu\Delta^\rho + \frac{1}{\epsilon}\Big(\frac{1}{12}\partial_\mu\partial_\nu D_{\rho\sigma} - \frac{1}{24}\Gamma_{\mu\lambda\nu}\Gamma_\rho{}^\lambda{}_\sigma\Big)\Delta^\mu\Delta^\nu\Delta^\rho\Delta^\sigma\bigg)_{(\boldsymbol{q},t)} + \mathrm{O}(\epsilon^{3/2}), \tag{6}$$

where $\Gamma_\mu{}^\nu{}_\rho$ are Christoffel symbols. The relation between the determinants in (3) and (4) is

$$\begin{aligned} D(\boldsymbol{q}+\Delta, t+\epsilon/2)^{-1/2} &= D(\boldsymbol{q}, t)^{-1/2}\exp[-\tfrac{1}{2}\partial_\mu D\Delta^\mu/D - \epsilon/4D\,\partial_t D \\ &\quad + \tfrac{1}{4}(\partial_\mu D\partial_\nu D/D - \partial_\mu\partial_\nu D/D)\Delta^\mu\Delta^\nu + \mathrm{O}(\epsilon^{3/2})]_{(\boldsymbol{q},t)} \end{aligned} \tag{7}$$

We can now replace in P_2 the action (sum of (5) and (6)) and the determinant using (7). The result is of the form $P_2(\boldsymbol{q}+\Delta, t+\epsilon; \boldsymbol{q}, t) = ND(\boldsymbol{q}, t)^{-1/2}\exp[-\tfrac{1}{2}\epsilon^{-1}D_{\mu\nu}(\boldsymbol{q}, t)\Delta^\mu\Delta^\nu + B]$, where B has terms in $\Delta, \Delta^2, \Delta^3/\epsilon, \Delta^4/\epsilon$, with coefficients evaluated at $(\boldsymbol{q}, t)$. One now develops $\exp[B]$ up to $\mathrm{O}(\epsilon)$; this generates extra terms in Δ^2, Δ^4/ϵ and $\Delta^6/\epsilon^2 = \mathrm{O}(\epsilon)$. We recall the equivalence rules (Dekker 1978, De Witt 1957, Langouche *et al* 1979b, Weiss 1978) (the symbol $\doteq$ stands for equivalence under the n-dimensional integral) $\Delta^\mu\Delta^\nu \doteq \epsilon D^{\mu\nu}$, $\Delta^\mu\Delta^\nu\Delta^\rho \doteq \Delta^\mu D^{\nu\rho} + \Delta^\nu D^{\mu\rho} + \Delta^\rho D^{\mu\nu}$, $\Delta^\mu\Delta^\nu\Delta^\rho\Delta^\sigma \doteq \epsilon^2(D^{\mu\nu}D^{\rho\sigma} + D^{\mu\rho}D^{\nu\sigma} + D^{\mu\sigma}D^{\nu\rho})$, $\Delta^\mu\Delta^\nu\Delta^\rho\Delta^\sigma\Delta^\tau\Delta^\lambda \doteq \epsilon^3(D^{\mu\nu}D^{\rho\sigma}D^{\tau\lambda}) + \text{perm})$, where all the functions $D^{\mu\nu}$ are evaluated at $(\boldsymbol{q}, t)$. Using these rules one transforms the terms in Δ^3/ϵ in terms in Δ, and the terms in $\Delta^2, \Delta^4/\epsilon, \Delta^6/\epsilon^2$ in terms in ϵ. We can now compare this final expression of P_2 with that of P_1 obtained from (3) and written as $ND^{-1/2}\exp(-\tfrac{1}{2}D_{\mu\nu}\Delta^\mu\Delta^\nu)$ $(1 - D_{\mu\nu}A^\mu\Delta^\nu + \epsilon V + \mathrm{O}(\epsilon^{3/2}))$; this determines $a^{\mathrm{F}}_\mu(\boldsymbol{q}, t)$ and $b^{\mathrm{F}}(\boldsymbol{q}, t)$, and L^{F} is

$$L^{\mathrm{F}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \tfrac{1}{2}D_{\mu\nu}(\dot{q}^\mu + a^\mu)(\dot{q}^\nu + a^\nu) - \tfrac{1}{2}\sqrt{D}\partial_\mu a^\mu/\sqrt{D} + \tfrac{1}{6}R - V, \tag{8}$$

where R is the curvature of the metric $D_{\mu\nu}(\boldsymbol{q}, t)$ and $a^\mu = A^\mu + \tfrac{1}{2}\sqrt{D}\partial_\nu D^{\mu\nu}/\sqrt{D}$. Note that L^{F} is a scalar, but this was of course implicit in the definition (4). We would like to point out that a calculation on similar lines to the one here has been made by Dowker (1974), where the comparison with a standard simple discretisation of reference (e.g. the prepoint one) is replaced by the explicit computation of the jump moments $\langle\Delta^\mu\Delta^\nu\rangle$ and $\langle\Delta^\mu\rangle$, and then the use of the standard Kolmogorov derivation of the diffusion equation. The calculation in Dowker (1974) has the nice feature of being covariant and simple through use of Ruse's Taylor series for the expansion of the action, but needs knowledge of this technique.

Formula (8) was recently obtained by a similar method by Weiss (1978); however, there the author finds it necessary to calculate the exact short-time propagator, i.e. $\langle \boldsymbol{q}_{j+1}|U(t_{j+1}, t_j)|\boldsymbol{q}_j\rangle$, in powers of ϵ and $\boldsymbol{\Delta}_j$ before using the equivalence relations mentioned above. This long calculation requires the scaling method of Graham

(1977a), and is unnecessary, since all the extra terms with respect to the prepoint discretisation we use vanish as we have shown in detail (Langouche *et al* 1979b); in fact, the method of Graham (1977a) provides a direct proof that only terms up to order ϵ are needed in intermediate steps. We remark also that recently Dekker (1979) has given a definition of the functional integral which consists of stating that the short-time propagator is to be computed by expanding the paths in Fourier series and integrating over the Fourier components. The result is (8), but in fact the proof given in Dekker (1978) is only valid in flat space since the use of local Euclidean frames is not allowed in finite regions of curved space and extra contributions will arise. Moreover, one can easily check in the simplest possible case (the harmonic oscillator) that the transformation to Fourier components in the integral $\int L(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \mathrm{d}t$ leads to a multiple integral which vanishes in the limit (Chang 1977). A way out of this difficulty can be to leave the normalisation unspecified, a prescription that works for the harmonic oscillator, but this deserves further study; or else to use a smoothed differentiation technique in the Fourier series, as mentioned in the earlier work of Dekker (ref. quoted in Dekker 1978), but this turns out to be equivalent to using a discretisation prescription, which is what the author is trying to avoid in Dekker (1978, 1979). The result (8) is not the Lagrangian L^{G} of Graham (1977a), which has $\frac{1}{12}R$ instead of $\frac{1}{6}R$ here. We conjecture that the L^{G} proposed by Graham is the one that determines the most probable path in the sense of Langouche *et al* (1978). The reason is that the WKB approximation $\sim[\det(-\delta^2\mathscr{A}^{\mathrm{G}}/\delta q^{\mu}_{j+1}\delta q^{\nu}_{j})]^{1/2} \times \exp[-\mathscr{A}^{\mathrm{G}}]$, $\mathscr{A}^{\mathrm{G}} = \int L^{\mathrm{G}}\, \mathrm{d}\tau$, is exact to $\mathrm{O}(\epsilon)$ for the short-time propagator.

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